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# FURTHER IMPROVEMENTS OF A NEW MODEL FOR TURBULENT CONVECTION IN STARS

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## ABSTRACT

In this paper we analyze the effect of several improvements of the input physics of a recent model for stellar turbulent convection. We first study the effect of (1) the inclusion of a variable molecular weight (Cox & Giuli's variable $Q$) and (2) the use of the newest opacities of Rogers & Iglesias. On the basis of the evolutionary tracks for the Sun (Fig. 2, curves 1 and 2), we conclude that the original model for turbulence with the mixing length $\Lambda = z$, together with $Q \neq 1$ and the new opacities, yields a fit to the solar $T_{\rm eff}$ within 0.5%. The model has no adjustable parameters.

Second, we propose a formulation of the mixing length $\Lambda$ that extends the purely nonlocal $\Lambda = z$ expression used in our previous work so as to include local effects. We derive the expression $\Lambda = \alpha(S, a)z$, where $S = 160A^2(\nabla - \nabla_{\rm ad})$ and where $1 - a$ represents the weight of local effects. The new expression generalizes both the mixing-length theory (MLT) phenomenological expression $\Lambda = \alpha H_p$, as well as our model $\Lambda = z$. By adjusting the parameter $a$ we can achieve an even better fit to the solar $T_{\rm eff}$. However, we also conclude that as long as one is interested in evolutionary studies only, the new version of the mixing length is not quite necessary since the previous model has no free parameters and yet yields a fit to $T_{\rm eff}$ within 0.5%. We point out, however, that this new model for $\Lambda$ may become instrumental in helioseismology where one needs to fit the solar radius considerably more accurately than in evolutionary studies.

Finally, in § 7 we use qualitative and quantitative arguments to discuss the new model versus the standard MLT and give reasons why we believe that the MLT should be abandoned.

*Subject headings:* convection — stars: interiors — Sun: interior — turbulence

## 1. INTRODUCTION

Recently the authors (Canuto & Mazzitelli 1991, hereafter CM) have suggested a new model to treat turbulent convection in stars. The distinguishing feature of this model is that it is no longer a one-eddy model like the mixing-length theory (MLT), for it includes the full spectrum of turbulent eddies. To account for the nonlinear interactions that yield such a spectrum, CM employed the EDQNM turbulence model which has been thoroughly tested in a variety of turbulence problems (Lesieur 1990).

Among the advantages of the CM model is that it contains no adjustable parameters since the mixing length is taken to be the geometrical depth inside the convective zone. The absence of free parameters obviously requires that the modeling of all other physical processes be as accurate as possible, which is not methodologically feasible with the MLT in which improvements on the input physics are somewhat a moot point since they can conceivably be represented by a different value of $\alpha$. Under this new approach, we discuss two improvements to the original CM model, namely the following:

1. Variation of the molecular weight within the convective zone, which was neglected in CM; and
2. A new set of radiative opacities.

By inserting these into the original CM model, we find that the observed solar $T_{\rm eff}$ is reproduced within 0.5%.

In the second part of the paper, we extend the CM model for $\Lambda$, which took into account only nonlocal effects, so as to include also local effects (defined more precisely in § 5.2) represented by a parameter $a$. The new relation is $\Lambda = \alpha(S, a)z$ (eq. [9]); for $a = 1$, we recover the CM expression $\Lambda = z$. By adjusting the local versus nonlocal contribution to $\Lambda$, we can achieve a fit to the solar $T_{\rm eff}$ even better than 5%, but we conclude that it represents an unnecessary precision, at least as far as evolutionary studies are concerned. This more sophisticated model may, however, become very useful in helioseismology where the requirements of accuracy in the prediction of the solar $T_{\rm eff}$ and thus of the solar radius far exceed that of interest in evolutionary studies.

Thus, we suggest that the original CM model with the two above additions is the model to be used in evolutionary studies for it contains no free parameters and fits the Sun very well.

In the last part of the paper, we discuss the MLT and specifically the basic assumption that the Sun-tuned value of the parameter $\alpha$ could meaningfully be used in other stars. Since the calibration of $\alpha$ depends on the low-$T$ opacities, which are more uncertain for metal-rich than for metal-poor mixtures, the tuning of $\alpha$ on a metal-rich star like the Sun is inadequate for metal-poor stars like globular clusters. We then introduce the concept of a *dilution function* of the overadiabatic gradient and show that its widely different shape in the Sun and other stars renders the basic MLT assumption untenable.

Our final conclusion is that the CM model should be preferred to the MLT *even if* the former did not fit the Sun with the same accuracy of the MLT. In fact, not only is the physical description of the temperature and density inversions in the convective zone more sound in the CM than in the MLT model, but within the CM model improvements on the input physics (low-$T$ opacities, nonideal gas thermodynamics, etc.) will not be masked by the resetting of adjustable parameters.

## 2. VARIABLE MOLECULAR WEIGHT

According to Cox & Giuli (1968, eq. [14.70], hereafter CG), the quantity $A$ entering the definition of the convective efficiency can be written as

$$A = Q^{1/2}(c_p \kappa \rho^2 \Lambda^2)(12acT^3)^{-1}(g/2H_p)^{1/2} , \quad (1)$$

with $Q$ given by

$$Q = (4 - 3\beta)\beta^{-1} - (\partial \ln \mu / \partial \ln T)_p , \quad (2)$$

where $\beta = P(\text{gas})/P(\text{total})$ accounts for the effect of radiation pressure.

In most stellar computations, it is customary to set $Q = 1$, as was done in CM, since the evaluation of the derivative term in equation (2) is not only time consuming, but can also cause numerical instabilities in the resulting values of the overadiabatic gradient. On the other hand, since in the solar subatmosphere $Q = 1.4$–$1.5$, even if the contribution of radiation pressure is negligible, a consistent application of the CM model requires the inclusion of $Q$. It is worth recalling that the only variation of molecular weight we are interested in is the one due to molecular dissociation and atomic ionization at constant elemental distribution, and not the more general one entering the definition of the Brunt-Vaisala frequency which accounts for gradients in the concentration of elements caused by nuclear reactions and/or gravitational settling (Zahn 1991). To further clarify this point, we provide a derivation of how to include a variable $\mu$ (see the Appendix).

## 3. NEW OPACITIES

We have employed the latest radiative opacities derived by F. J. Rogers & C. A. Iglesias (1991, private communication), supplemented by the ATLAS values for $T < 6000$ K (Kurucz 1991). For both cases, the metal mixture is taken from Anders & Grevesse (1989). Although we have used the ATLAS values only in the optical atmosphere, we did check that in the range $6000 \text{ K} < T < 10{,}000$ K, the two sets of opacities merge smoothly. This is, however, not so for other low-$T$ opacities available in the literature: for example, according to our tests, D. R. Alexander's opacities (1990, private communication) for $T < 1000$ K yield a $T_{\text{eff}}$ about 2% larger than our value, while the opacities by Weiss et al. (1991, metal mixture WKM10) yield a $T_{\text{eff}}$ about 2% lower than ours.

## 4. THE COMPLETE SOLAR MODEL

Although the evolutionary code employed in this paper has been described in detail in Mazzitelli (1989), it is important to recall some of the main features:

1. Thermodynamics: The equation of state for a nonideal gas, with variable degree of ionization, is taken from the work of Magni & Mazzitelli (1979).
2. Opacity: See § 3.
3. Optical atmosphere: A simple $T(\tau)$ relation (Henyey, Vardya, & Bodenheimer 1965) was adopted all the way down to $\tau = \frac{2}{3}$ or to the onset of convection.
4. The nuclear network accounts for the following elements:

$$^1\text{H},\ ^3\text{He},\ ^4\text{He},\ ^{12}\text{C},\ ^{13}\text{C},\ ^{13}\text{N},\ ^{14}\text{N},\ ^{15}\text{N},\ ^{16}\text{O},\ ^{17}\text{O} ,$$

and the corresponding 15 nuclear reactions of the largest cross sections; we have also added the beryllium branching and the $^{13}$N decay, according to Fowler, Caughlan, & Zimmerman (1975) and Harris et al. (1983). The distribution of the CNO elements follows Ross & Aller (1976), while the total metal abundance $Z$ was taken equal to 0.018.
5. Numerical features: The code performs a straightforward Newton-Raphson integration up to the base of the optical atmosphere with the mass $m(r)$ as the independent variable. The thermodynamics tables are linearly interpolated in the variables $\log T$, $\log \rho$, and $Y$ (via the additive volume law), using five tables of different He content; for the opacities, cubic spline functions are used to interpolate in $\log T$ and $\log \rho$, and quadratic in $\log Z$ (metal content) and $Y$.

The optical atmosphere is integrated using the optical depth as an independent variable (about 100 mesh points), and the number of internal mesh points varies between 600 and 700. To reach the present solar luminosity, the evolution takes about 200 time steps of different duration depending on the specific phase. The chemical evolution scheme in the radiative regions follows the linearization procedure of Arnett & Truran (1969) and a zero-order Runge-Kutta scheme in the rapidly vanishing convective core.

With the above numerical scheme and input data, the initial helium abundance $Y$ required to obtain the present solar luminosity of $3.846 \times 10^{33}$ ergs s$^{-1}$, at an age of 4.6 Gyr (an average of the values given by Guenther 1989, Bahcall & Ulrich 1988, and Demarque, Guenther, & Van Altena 1986) turns out to be $Y = 0.287$.

## 5. THE MIXING LENGTH $\Lambda$

### 5.1. *The CM Nonlocal Model* $\Lambda = z$

As discussed in CM, a treatment of compressible turbulent convection would provide *both* the convective flux and a unit of length. The latter is expected to be the pressure scale height, and physical arguments in support of that view were presented in CM. On the other hand, if one treats turbulence as incompressible, the problem does not provide a unit of length, with the implication that the mixing length $\Lambda$ cannot be determined from within the formalism itself. In recognition of the fact that in fully compressible turbulence, $H_p$ would be the natural unit of length, one writes $\Lambda = \alpha H_p$ and then fixes $\alpha$ using stellar data.

In CM we proposed a model for $\Lambda$ based on a physical analogy with the case of mechanically driven turbulence. In that case, only relatively large wavenumbers become unstable which correspond to small-eddy turbulence; large eddies are absent. The lack of large structures implies that the ensuing turbulence is an essentially local phenomenon, and the corresponding mixing length depends more on local variables than on the nonlocal ones. This is no longer the case for thermally driven turbulence (convection) in which both large and small eddies are present. In this case $\Lambda$ must be written as

$$\Lambda = \Lambda(\ell, z) , \quad (3)$$

where *the length $\ell$ represents local variables and z represents the nonlocal one.* In CM it was suggested that since in the case of convection the nonlocal dependence is more relevant (large eddies carry most of the flux), equation (1) may be approximated by

$$\Lambda \approx \Lambda(z) , \quad (4)$$

of which the simplest form is

$$\Lambda = z \, , \tag{5}$$

which is the form used in CM. Equation (5) accounts for the phenomenon of vertical stacking of the eddies, has no free parameters, yields a reasonable fit to the Sun's data, and yet is not fully satisfactory for it accounts for nonlocal effects only.

### 5.2. *How to Include Local Effects*

We endeavor to construct a more complete model for $\Lambda$ so as to account for both the $z$- and $\ell$-dependence. This confronts us with two new problems: *first* we must construct an expression for the local length $\ell$, and *second* we must construct the function (3). Before we do so, we notice that since we use the incompressible model of turbulence developed in CM to compute $\ell$, the result will be of the form

$$\ell/\Lambda = f(S) \, , \tag{6}$$

that is, the local length $\ell$ will be given in units of the only length that appears in the problem, the unknown mixing length $\Lambda$. Equation (3) thus becomes an implicit relation for $\Lambda$. We shall use the CM model to compute the values of the dimensionless function $f(S)$ versus $S$, where

$$S = 162A^2(\nabla - \nabla_{\rm ad}) \, . \tag{7}$$

Equation (6) will be derived in § 5.3 and equation (7) in the Appendix.

Given these premises, let us first construct the function $\Lambda$. To do so, we begin by suggesting that the relevant physical variables are not $\Lambda$, $\ell$, and $z$ themselves but rather $\ln \Lambda$, $\ln \ell$, $\ln z$. The model we suggest consists of taking $\ln \Lambda$ as a linear combination of the two other variables,

$$\ln \Lambda = a \ln z + b \ln \ell \tag{8a}$$

or

$$\Lambda = z^a \ell^b \, , \tag{8b}$$

with $a = 1 - b$. Substituting equation (6) for $\ell$, we obtain

$$\Lambda = \alpha(S, a) z \, , \tag{9a}$$

where

$$\alpha(S, a) = f(S)^{(1-a)/a} \, . \tag{9b}$$

Equations (9) represent the new formula for $\Lambda$. It may be noticed that since for a polytrope $H_p = p/\rho g \sim z$, equation (9a) can also be viewed as a generalization of the standard relation

$$\Lambda = \alpha H_p \tag{10}$$

with the important difference that $\alpha$ is no longer an unknown quantity but a *variable* that we shall compute. The CM model, $\Lambda = z$, corresponds to $a = 1$.

### 5.3. *Evaluation of the Function $f(S)$*

We suggest that $\ell$ be identified with the average over all wavenumbers of the mean free path $\lambda(k)$ of an eddy of arbitrary wavenumber $k$,

$$\ell = \frac{\int \lambda(k) G(k) dk}{\int G(k) dk} \tag{11}$$

with $G(k)$ taken to be the convective flux, that is,

$$G(k) \equiv E(k)[n_s(k) + \nu k^2] \, , \tag{12}$$

where $n_s(k)$ is the rate at which the thermal instability injects energy into the flow (CM, eq. [18]) and $E(k)$ is the turbulent energy spectrum solution of the nonlinear equation (20) of CM. Graphical representations of $E(k)$ can be found in Figures 1–4 of CM. The integrals in equation (11) extend over all wavenumbers, that is, one integrates over eddies of all sizes. It remains to define the mean free path $\lambda(k)$. Since the choice is not unique, we have investigated two alternatives which fortunately yield comparable results. The first choice,

$$\lambda(k) = 2\pi/k \, , \tag{13}$$

identifies $\lambda(k)$ with the size of an eddy.

Alternatively, we may write

$$\lambda(k) = u(k)\tau(k) \, , \tag{14}$$

where $\tau(k)$ is the lifetime of an eddy and $(1/2)u(k)^2$ is the kinetic energy of a band of wavenumbers centered around the wavenumber $k$, that is,

$$\tfrac{1}{2}u(k)^2 = \int_0^k E(p) dp \, . \tag{15}$$

To evaluate $\tau(k)$, we may identify it with the correlation time scale governed by the nonlocal interactions which, as explained in CM, are primarily determined by a turbulent viscosity, $\nu_t(k)$, representing the effect of all the eddies smaller than $k^{-1}$ on it. In that case,

$$\tau(k)^{-1} = k^2[\nu + \nu_t(k)] \, , \tag{16a}$$

where

$$\nu_t(k) = (\tfrac{1}{30}) \int_k^\infty \tau(p)[5E(p) + pE'(p)] dp \, . \tag{16b}$$

Substituting equation (16a), we obtain

$$\nu_t(k) = \left\{ \tfrac{1}{15} \int_k^\infty p^{-2}[5E(p) + pE'(p)] dp \right\}^{1/2} - \nu \, , \tag{16c}$$

which expresses $\nu_t(k)$ entirely in terms of the energy spectral function $E(k)$. Finally, using equations (15) and (16a), we have from equation (14)

$$\lambda(k) = \frac{[2 \int_0^k E(p) dp]^{1/2}}{k^2[\nu + \nu_t(k)]} \, . \tag{17}$$

Using the formalism developed in CM and the numerical results for $E(k)$ obtained there, we have evaluated equation (11) and written the result as in equation (6). The resulting dimensionless functions $f(S)$ versus $S$ are given in Table 1 and Figure 1.

TABLE 1

VALUES OF THE FUNCTION $f(S)$ VERSUS $S$[a]

| $\log S$ | $f(S)$ | $f(S)$ |
|---|---|---|
| 2............. | 1.7558 | 1.0994 |
| 4............. | 0.9318 | 0.9202 |
| 6............. | 0.7106 | 0.7544 |
| 8............. | 0.5954 | 0.6638 |
| 10............. | 0.5446 | 0.6190 |
| 12............. | 0.5204 | 0.5958 |
| 14............. | 0.5078 | 0.5778 |
| 16............. | 0.5124 | 0.5860 |

[a] The two sets of values of $f(S)$ correspond to the average of equations (13) and (14) over the convective flux.

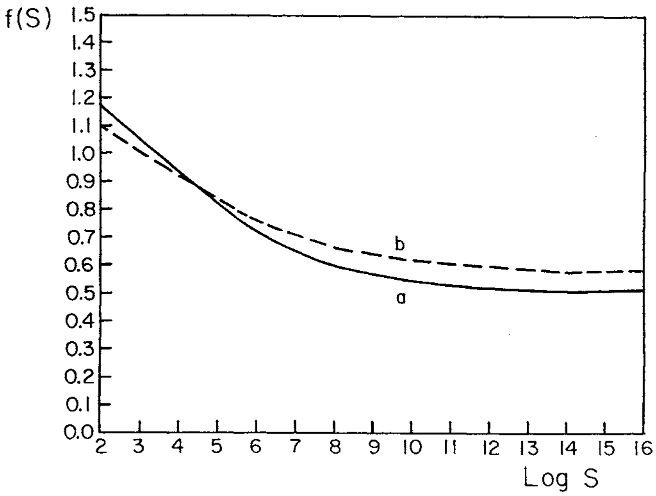


FIG. 1.—The function $f(S)$ vs. $S$. Curves *a* and *b* correspond to expressions (13) and (17) in eq. (11). See also Table 1.

## 6. RESULTS

To appreciate the changes brought about by each of the changes, we shall proceed as follows:

1. *New opacities only* (CM model, $\Lambda = z$, $Q = 1$).—The resulting evolutionary track is presented in Figure 2 (curve 1). The fit is not fully satisfactory for it yields a value for $T_{\rm eff} \sim 2\%$ lower than the observed value. The difference with the result presented in CM is due to the new opacities which are larger than D. R. Alexander's (1990, private communication) by up to a factor of 2.

2. *New opacities and* $Q \neq 1$ (CM model, $\Lambda = z$).—The new track is represented by curve 2 of Figure 2: the predicted $T_{\rm eff}$ is 0.5% lower than the observed value. The increase in $T_{\rm eff}$ can easily be explained: in regions of partial ionization, $Q$ is always larger than 1 since the molecular weight increases as the temperature decreases; in addition, since the numerical value of the variable $A$ is affected in the same way, the overadiabatic excess is consequently reduced, leading to a larger value of $T_{\rm eff}$ (the larger the value of $A$, the lower the adiabaticity).

3. *New opacities,* $Q \neq 1$ *and new mixing length,* $\Lambda = \alpha(S, a)z$.—As a first step, we verified that the two models for $f(S)$ described in § 5 yield almost identical results. We then tested two cases: $a = 2$ and 0.7, curves 3 and 4 of Figure 2, respectively. Although the resulting difference in $T_{\rm eff}$ is rather small ($\sim 0.5$), some conclusions can be reached. Given the behavior of $f(S)$ versus $S$ (Table 1) and equations (9), choosing $a < 1$ leads to $\Lambda > z$ for layers close to the surface of the star and $\Lambda < z$ for the deep layers; the opposite attains if one chooses $a > 1$. Let us consider the latter case. As already discussed, the lower the value of $A$ ($\sim \Lambda^2$), the larger the overadiabatic excess $\nabla - \nabla_{\rm ad}$. In this light, the choice $a > 1$ leads to a taller peak of $\nabla - \nabla_{\rm ad}$ at the top of the CZ, as well as to a faster decay (with respect to the $a = 1$ case). Since this model fits the Sun better than the other we have tried, with this test we have gained valuable information for future development of the theory.

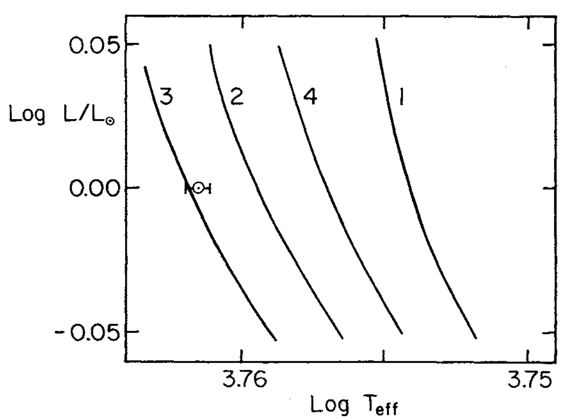


FIG. 2.—A blowup of the theoretical HR diagram in the vicinity of the Sun, marked with the solar symbol, and with the observational error bar in log $T_{\rm eff}$. Curve 1 is the track according to CM, $\Lambda = z$, $Q = 1$, and with chemical and numerical updates in the code; curve 2, $\Lambda = z$, $Q \neq 1$; curve 3, $\Lambda = \alpha(S)z$, $a = 2$, and $Q \neq 1$; curve 4, $\Lambda = \alpha(S)z$, $a = 0.7$, and $Q \neq 1$.

In conclusion, we note the following the original CM model with $\Lambda = z$, supplemented by $Q \neq 1$ and the new opacities, fits the Sun within the overall theoretical uncertainties. Thus, we recommend this model when computing stellar structures and evolutionary tracks for general purposes. By the same argument, we strongly discourage the tuning of the parameter *a other than* for numerical experiments on the theory of convection as such (as we have done above) or for very few specific tasks, like the construction of models for helioseismology.

## 7. WHY THE MLT SHOULD BE REPLACED

The lack of a viable alternative to the MLT, and the lure of both its simplicity and its apparent ability to fit the $T_{\rm eff}$ of almost any star by the simple device of adjusting a single free parameter, have undoubtedly played a significant role in its widespread acceptance in evolutionary studies. Over the years, the conviction was created that one can carry out meaningful comparisons between theory and observations, and thus arrive at seemingly precise quantitative conclusions based on differences in $T_{\rm eff}$ of the order of a very few percent. For example, we believe that the age of globular clusters is also affected by the model of turbulent convection that one adopts, even though this is by no means the only factor since element distribution, gravitational and thermal diffusion of helium, etc., can play an equally important role. Before any reasonably sound claim can be made, one must therefore analyze many interacting factors, and this requires a detailed comparison between theory and observations.

To justify the contention that the MLT be replaced by a new model, we put forward qualitative and quantitative arguments suggesting that the standard procedure of tuning $\alpha$ on the Sun and then using the result for other stars is invalid.

The *qualitative argument* stems from considerations about opacities. As we know, the low-temperature radiative opacities are far from being known with the required accuracy. More importantly, it is expected that the uncertainties in the opacities be a function of the chemical composition. In fact, in metal-rich mixtures, atomic and molecular transitions dominate, as do the few free electrons (due to metals at low temperature), through the appearance of $H^-$, etc. For metal-poor mixtures, even if the problems are in principle the same, the global effect of metals on the quantitative evaluation of the opacities is smaller, so that the uncertainties in the low-temperature opacities are expected to be larger for metal-rich than for metal-poor mixtures.

This leads us to conclude that an $\alpha$ tuned on a metal-rich star like the Sun cannot, in principle, be used to treat stars that are metal poor, for example, globular clusters, since the Sun-tuned MLT resets also the uncertainties in the solar opacities, which are different from the uncertainties in the opacities of metal-poor stars. Moreover, the solar tuning of $\alpha$ entails a narrow range of $T$ and $\rho$, mainly those of the solar subatmosphere, which are different from those found in the subatmospheres of other stars (e.g., red giants). Here again, since different ranges of $T$ and $\rho$ can introduce different uncertainties in the opacities, the Sun-tuned values may not be consistent with the physics of these stars. Similar arguments have been put forward before (VandenBerg 1983, 1985; Pedersen, VandenBerg, & Irwin 1990). However, because of the lack of a viable alternative to the MLT, almost all the comparisons between theory and observations have been based on the Sun-tuned values of $\alpha$ (VandenBerg 1985; VandenBerg & Bell 1986; Chieffi & Straniero 1989; Sarajedini & Demarque 1990).

The *semiquantitative argument* stems from the fact that the scale length adopted in the MLT is far wider than the one expected on physical grounds. In other words, in the MLT one uses a scale length of the order of $2H_p$ to compute the local value of the overadiabatic gradient which, according to the same MLT, rises steeply, reaches a narrow maximum, and rapidly decays in a few tenths of $H_p$. Although we cannot offer a mathematical proof, it is reasonable to expect that the use of a manifestly incorrect and exceedingly large $\Lambda$ (so that the real phenomenon is forced to occur entirely within a small fraction of $\Lambda$), should lead to an artificial *spreading and dilution* of the peak of overadiabaticity. Indeed, inspection of Figure 8 of CM reveals that the overadiabatic peak is definitively shallower in the MLT than in the CM model.

The physical justification behind this different behavior is twofold. The first reason lies in the different convective fluxes $F_c$ in the MLT versus the CM models: at low $S$, close to the surface, $F_c(\mathrm{CM}) < F_c(\mathrm{MLT})$, whereas for large $S$, $F_c(\mathrm{CM}) \sim 10F_c(\mathrm{MLT})$. In the upper convective layers this requires that the CM overadiabaticity be larger than MLT. This conclusion is clearly the result of comparing two models: it does not prove that use of a more correct $\Lambda$ should give rise to a sharper overadiabatic peak.

The proof of the last statement comes from considering the second argument: the lower the value of $A$ in equation (1), the lower the convective flux is and the larger the overadiabaticity has to be. Since $A$ is proportional to $\Lambda^2$, as one moves from, say, $2H_p$ to (smaller) values, of the same order of magnitude of the scale length characterizing the rise and fall of the overadiabatic peak, the value of $A$ decreases of orders of magnitude, and the overadiabaticity peak becomes much taller also in the MLT framework.

This argument is independent of the CM model and shows that use of a more physically correct mixing length would in any case predict overadiabatic peaks much higher than what is currently derived from standard stellar models, that is, MLT with $\Lambda = \alpha H_p$.

Of course, even if the above argument is agreed upon, one could still adopt the restricted viewpoint that getting the correct value of $T_{\mathrm{eff}}$ is already an achievement (considering the crudeness of the MLT), even if one is forced to give up a credible description of the actual behavior of the temperature profile in the subatmosphere. In other words, one might be willing to accept the unreliable MLT local values of temperature in the subatmosphere so long as the integral of the temperature profile along the subatmosphere yields the correct value of $T_{\mathrm{eff}}$.

This, indeed, seems to have been the prevailing attitude toward the MLT. However, not only have recent studies on pulsational properties of stars stressed the importance of a reliable determination of the *internal temperature profiles*, but it is also our contention that for this last attitude toward the MLT to be acceptable, a further requirement has to be met: *the "dilution function" of the overadiabatic gradient, once tuned to the Sun, must be the same for any other star.*

A detailed knowledge of the "dilution function" is fortunately not required since at every point along the subatmosphere, we expect it to be a function of the ratio between the overall scale length adopted in the theory and the geometrical scale length according to which the physical mechanism evolves locally. It will then be a function of the ratio $R = \alpha H_p/z$, where $z$ is the geometrical distance from the top of convection. To quantify the argument, we used the MLT to compute two models: the Sun and a star of 0.8 $M_\odot$, with $Y = 0.23$, $Z = 0.001$. In the HR diagram, these stars have almost the same position, and the latter is representative of stars at the turnoff in globular clusters. We adopted the Los Alamos (Weiss, Keady, & Magee 1991) low-temperature opacities so that in order to fit the Sun, we must have $\alpha = 2$. Other choices of opacity and $\alpha$ would not change the gist of our argument.

The behavior of the convective gradients and of $R$ for the two stars is shown in Figure 3. For the Sun, the peak of overadiabaticity is reached in a region where the value of the ratio $R$ is about 7; for the 0.8 $M_\odot$ star, the value of $R$ at peak of overadiabaticity is instead about 13, twice as large as for the Sun. Note that we would have found a large difference between the two values of $R$ even if we had adopted two slightly different tunings of $\alpha$ for the Sun and for a globular cluster star.

The conclusion seems unavoidable: the "dilution functions" for the Sun and for the 0.8 $M_\odot$ star are profoundly different. Thus, there is no justification in assuming that if the integral of the "diluted" temperature profile yields the correct value of $T_{\mathrm{eff}}(\mathrm{Sun})$, the same will be true for a 0.8 $M_\odot$ star.

We have one more piece of evidence in support of our conclusion: if the MLT value of $R$ at the peak of overadiabaticity is larger for the 0.8 $M_\odot$ star than for the Sun, one could legitimately expect that the overadiabaticity profile be "diluted" for

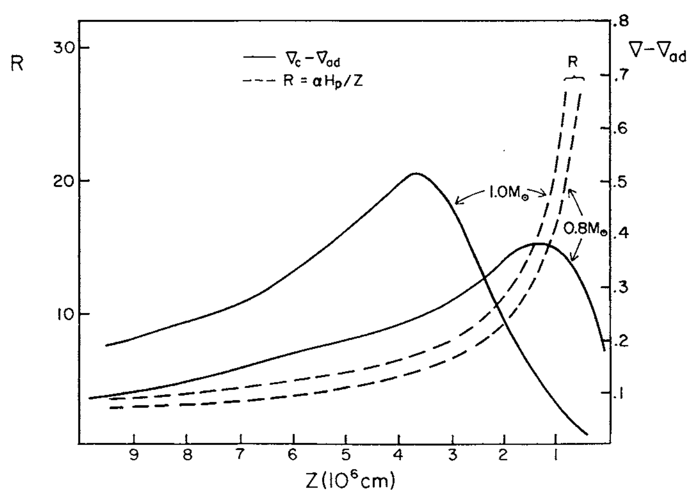


FIG. 3.—The behavior of the overadiabatic excess in the subatmospheres of the Sun and of a star of 0.8 $M_\odot$, $Y = 0.23$, $Z = 0.001$, computed according to the MLT. The ratios $R = \alpha H_p/z$ are also plotted, showing that, at the peak of overadiabaticity, $R$ for the Sun is about half the $R$ for the 0.8 $M_\odot$ star.

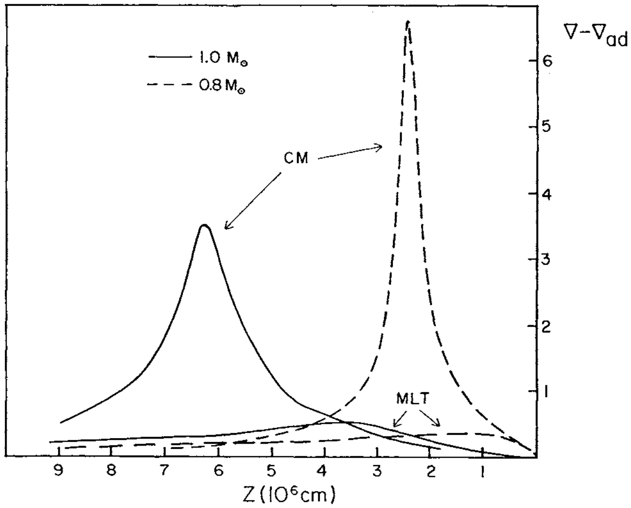


FIG. 4.—The behavior of the overadiabatic excess for the Sun and the 0.8 $M_\odot$ star. In the CM model, the values are about an order of magnitude larger than in the MLT case. Contrary to what occurs in the MLT, the peak for the 0.8 $M_\odot$ star is a factor of 2 larger than for the Sun.

the 0.8 $M_\odot$ more than for the Sun. In fact, within the MLT framework, the maximum value of the overadiabatic gradient is, for the 0.8 $M_\odot$, slightly lower than for the Sun; in the CM model, we should then expect the reverse, a larger peak of overadiabaticity for the 0.8 $M_\odot$ star than for the Sun. This is indeed what is found: for the 0.8 $M_\odot$, the maximum value of the overadiabatic gradient is $\sim 6.0$, whereas that for the Sun is not even half that value (Fig. 4). We call attention to the fact that this is due more to the use of a smaller mixing length, that is, $\Lambda = z$ rather than $\Lambda = \alpha H_p$, than to the use of the CM model itself.

Our conclusion is that the "dilution function," once tuned to the Sun, is far from being adequate even for relatively similar stars, and this in turn seems to undermine the popular assumption that what is good for the Sun is good for other stars.

Furthermore, neither the MLT nor the CM models should be used in conjunction with $\Lambda = \alpha H_p$, although a test with the latter was performed in CM for illustrative purposes. This is so because with the $\Lambda = \alpha H_p$ model fitted to the Sun, the problem of the differences in the *spreading function* appears also in the CM model.

We are therefore left with only one version of the CM model: the convective flux given in CM together with $\Lambda = z$, $Q \neq 1$, and updated physical inputs, for example, the new opacities.

## 8. CONCLUSIONS

This paper contains the following conclusions:

1. We studied an improved version of the CM, which contains (a) $Q \neq 1$ and (b) new opacities. The new model was tested on the Sun and found to reproduce the observed $T_{eff}$ within 0.5%.

2. We suggest an extension of the model for the mixing length $\Lambda = z$ used in CM so as to account for local contributions. The new expression is $\Lambda = \alpha(S, a)z$. By adjusting the local contribution $1 - a$, one can in principle achieve a fit to the solar $T_{eff}$ better than 0.5%. However, we find that this is not really necessary, at least within the contest and the requirements of evolutionary studies, while it may become an important degree of freedom to be exploited in helioseismology where one needs a solar model that predicts the solar radius with an exceedingly high accuracy.

3. We put forward arguments to suggest that the MLT ought to be replaced by the new model. In fact, in addition to the more physical description of the spectrum of convective eddies by the CM model than by the MLT, the following new points have emerged:

(a) the MLT tends to hide errors in the low-temperature radiative opacities, giving results of *unpredictable reliability* when its tuning on the solar metal abundance is applied to stars of different metal abundance;

(b) the CM theory keeps separate (and separately improvable) the uncertainties due to the opacities and to convection as a whole;

(c) the MLT gives incorrect values for the local overadiabatic gradients and temperatures in the subatmosphere, since the scale length involved is inadequate by an order of magnitude;

(d) the CM theory takes into account the expected physical behavior of the scale length, giving more reliable gradients, as required for pulsational computations.

The CM model has improved a serious shortcoming of the MLT model, namely the inadequate representation of the full eddy spectrum. Other improvements remain to be incorporated, to wit, compressibility, inhomogeneities, overshooting, etc., and they are part of the work presently pursued by the authors.

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## APPENDIX

As discussed in CM, the convective flux $F_c$ can be written as

$$F_c = KTH_p^{-1}(\nabla - \nabla_{ad})\Phi(S)\,, \tag{A1}$$

with the dimensionless variable $S$ given by

$$S = g\alpha T(c_p\chi^2)^{-1}\left(-\frac{\partial\Sigma}{\partial z}\right)\Lambda^4\,, \tag{A2}$$

where $\Sigma$ is the entropy. All the other symbols are defined in CM. Different models of turbulent convection yield different forms of $\Phi$: the MLT expression is given in CM by equation (4), while the new CM model yields equations (32) and (33). In what follows, we shall compute equation (A2) in the general case of a variable molecular weight $\mu$. The general expression for $d\Sigma$ is (Batchelor 1970; Moffat 1977)

$$d\Sigma = c_p T^{-1} dT - \alpha V \, dP \, , \tag{A3}$$

where the thermal expansion coefficient $\alpha$ is defined as

$$\alpha = V^{-1}\left(\frac{\partial V}{\partial T}\right)_P . \tag{A4}$$

Using the thermodynamic identity

$$\left(\frac{\partial P}{\partial T}\right)_V = -\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P , \tag{A5}$$

one obtains

$$\alpha = T^{-1} \chi_T \chi_\rho^{-1} , \tag{A6}$$

where (see CG)

$$\chi_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho , \qquad \chi_\rho = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T . \tag{A7}$$

Equation (A6) generalizes the expression for $\alpha$ used in CM (eq. [44]).

Alternatively, using the CG symbol $Q$, we derive

$$\alpha T = 1 - \left(\frac{\partial \ln \mu}{\partial \ln T}\right)_P = 1 - \chi_T^\mu = Q . \tag{A8}$$

Thus, it follows that

$$\chi_T = \chi_\rho (1 - \chi_T^\mu) , \tag{A9}$$

and we derive

$$-\left(\frac{H_p}{c_p}\right)\frac{\partial \Sigma}{\partial z} = \nabla - \left(\frac{P}{\rho T c_p}\right)\frac{\chi_T}{\chi_\rho} . \tag{A10}$$

Using the relation (CG, eq. [9.86])

$$c_p - c_v = \left(\frac{P}{\rho T}\right)\chi_T^2 \chi_\rho^{-1} \tag{A11}$$

and the notation $\gamma \doteq c_p/c_v$, the right-hand side of equation (10) transforms to

$$\nabla - (1 - \gamma^{-1})\chi_T^{-1} . \tag{A12}$$

Using the definition of $\nabla_{ad} = (d \ln T / d \ln P)_{ad}$, it is straightforward to show that

$$\nabla_{ad} = (1 - \gamma^{-1})\chi_T^{-1} . \tag{A13}$$

Putting together these results, we have, making use of the variable $A$,

$$S = 162 A^2 (\nabla - \nabla_{ad}) \tag{A14}$$

with $A$ given by equation (1). It may be noticed that $\mu$ appears explicitly only in the quantity $Q$, although clearly $\nabla_{ad}$ is no longer equal to $1 - \gamma^{-1}$ as in the $\mu$ = constant case.

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